

ON EXOTIC MODULAR TENSOR CATEGORIES

SEUNG-MOON HONG, ERIC ROWELL, AND ZHENGHAN WANG

Dedicated to the memory of Xiao-Song Lin

ABSTRACT. It has been conjectured that every $(2+1)$ -TQFT is a Chern-Simons-Witten (CSW) theory labeled by a pair (G, λ) , where G is a compact Lie group, and $\lambda \in H^4(BG; \mathbb{Z})$ a cohomology class. We study two TQFTs constructed from Jones' subfactor theory which are believed to be counterexamples to this conjecture: one is the quantum double of the even sectors of the E_6 subfactor, and the other is the quantum double of the even sectors of the Haagerup subfactor. We cannot prove mathematically that the two TQFTs are indeed counterexamples because CSW TQFTs, while physically defined, are not yet mathematically constructed for every pair (G, λ) . The cases that are constructed mathematically include:

- (1) G is a finite group—the Dijkgraaf-Witten TQFTs;
- (2) G is torus T^n ;
- (3) G is a connected semi-simple Lie group—the Reshetikhin-Turaev TQFTs.

We prove that the two TQFTs are not among those mathematically constructed TQFTs or their direct products. Both TQFTs are of the Turaev-Viro type: quantum doubles of spherical tensor categories. We further prove that neither TQFT is a quantum double of a braided fusion category, and give evidence that neither is an orbifold or coset of TQFTs above. Moreover, representation of the braid groups from the half E_6 TQFT can be used to build universal topological quantum computers, and the same is expected for the Haagerup case.

1. INTRODUCTION

In his seminal paper [Witt1], E. Witten invented Chern-Simons $(2+1)$ -topological quantum field theory (TQFT), and discovered a relation between Chern-Simons TQFTs and Wess-Zumino-Novikov-Witten (WZW) conformal field theories (CFTs). To be more precise, CFTs here should be referred to as chiral CFTs, as opposed to full CFTs. The connection between Chern-Simons-Witten $(2+1)$ -TQFTs and WZW models has spawned an application of TQFT and rational CFT (RCFT) to condensed matter physics (see [RSW] and the references therein). In fractional quantum Hall liquids, Chern-Simons-Witten theories are used to describe emerged topological properties of the bulk electron liquids, whereas the corresponding CFTs describe the boundary physics of the Hall liquids (see [Wil] and

The first and third authors are partially supported by NSF FRG grant DMS-034772. The third author likes to thank F. Xu, M. Müger, V. Ostrik, and Y.-Z. Huang for helpful correspondence.

the references therein). A unifying theme in the mathematical formulation of both $(2+1)$ -TQFTs and CFTs is the notion of a modular tensor category (MTC) [Tu]. Modular tensor categories are the algebraic data that faithfully encode $(2+1)$ -TQFTs [Tu], and are used to describe anyonic properties of certain quantum systems (see [Ki2] [DFNSS] [Wa] and the references therein). In this paper, we will use the terms $(2+1)$ -TQFT, or just TQFT in the future, and MTC interchangeably (We warn readers that it is an open question whether or not TQFTs and MTCs are in one-one correspondence, see e.g. [BK]. But an MTC gives rise to a unique TQFT [Tu]). Our interest in MTCs comes from topological quantum computing by braiding non-abelian anyons in the sense of [FKLW] (cf. [Ki1]). From this perspective, we are interested in an abstract approach to MTCs free of algebraic structures such as vertex operator algebras (VOAs) or local conformal nets of von Neumann algebras, whose representation theory gives rise to MTCs (see [Hu][KLM][EK] and the references therein).

Known examples of MTCs that are realized by anyonic quantum systems in real materials are certain abelian MTCs encoding Witten's quantum Chern-Simons theories for abelian gauge groups at low levels (see [Wil]). The physical systems are 2-dimensional electron liquids immersed in strong perpendicular magnetic fields that exhibit the so-called fractional quantum Hall effect (FQHE). In these physical systems, the representations of the braid groups from the MTCs describe braiding statistics of the quasi-particles, which are neither bosons nor fermions. F. Wilczek named such exotic quasi-particles *anyons*. Confirmation of the realization of non-abelian MTCs in FQH liquids is pursued actively in experiments (see [DFNSS] and the references therein).

Inspired by FQHE, we may imagine that there are physical systems to realize many MTCs. With this possibility in mind, we are interested in the construction and classification of MTCs. Since TQFTs and CFTs are closely related to each other, we may expect all the known constructions of new CFTs from given CFTs such as coset, orbifold, and simple current extension can be translated into the TQFT side, and then to the MTC side in a purely categorical way. After many beautiful works, it seems that those constructions cannot in general be defined in the purely categorical setting. On the CFT side, it has been expressed several times in the literature that all known rational CFTs are covered by a single construction: Witten's quantum Chern-Simons theory. In particular, the following conjecture is stated in [MS1]:

Conjecture 1: *The modular functor of any unitary RCFT is equivalent to the modular functor of some Chern-Simons-Witten (CSW) theory defined by the pair (G, λ) with G a compact group and $\lambda \in H^4(BG; \mathbb{Z})$.*

Another conjecture, attributed to E. Witten [Witt3], was stated as Conjecture 3 in [MS1]:

Conjecture 3: *All three dimensional topological field theories are CSW theory for some appropriate (super)-group.*

CSW theories with compact Lie groups are written down in [DW], and they are labeled by a pair (G, λ) , where G is a compact Lie group, and $\lambda \in H^4(BG; \mathbb{Z})$. A modular functor is just the 2-dimensional part of a TQFT [Tu][MS2]; TQFTs from CSW theory as in the conjectures will be called **CSW** TQFTs. Therefore, we paraphrase the two conjectures as:

Conjecture CSW: *Every $(2+1)$ -TQFT is a CSW TQFT for some pair (G, λ) , where G is a compact Lie group, and $\lambda \in H^4(BG; \mathbb{Z})$ a cohomology class.*

Since TQFTs are faithfully encoded by MTCs [Tu], translated into the MTC side, this conjecture says that any unitary MTC is equivalent to one from some unitary CSW TQFT. If this conjecture holds, we will have a conceptual classification of $(2+1)$ -TQFTs. Of course even if the conjecture were true, to make such a classification into a mathematical theorem is still very difficult.

There are three families of compact Lie groups for which we have mathematical realizations of the corresponding $(2+1)$ CSW TQFTs:

- (1) G is finite [DW][FQ];
- (2) G is a torus T^n [Ma][BM];
- (3) G is a connected semi-simple Lie group [RT][Tu].

Given such an attractive picture, we are interested in the question whether or not all known TQFTs fit into this framework. An MTC or TQFT will be called *exotic* if it cannot be constructed from a CSW theory. In this paper, we will study two MTCs which seem to be exotic: the quantum doubles $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ of the spherical fusion categories \mathcal{E} and \mathcal{H} generated by the even sectors of the E_6 subfactor (a.k.a. $\frac{1}{2}E_6$), and the even sectors of the Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$ [AH]. Unfortunately we cannot prove that these two unitary MTCs are indeed exotic. The difficulty lies in describing mathematically all unitary CSW MTCs, in particular those from non-connected, non-simply-connected Lie groups G . When G is finite, the corresponding Dijkgraaf-Witten TQFTs are (twisted) quantum doubles of the group categories, which are well understood mathematically (see [BK]). When G is a torus, the corresponding TQFTs are abelian, and are classified in [BM]. When G is a connected semi-simple Lie group, the CSW MTCs are believed to correspond mathematically to MTCs constructed by N. Reshetikhin and V. Turaev based on the representation theory of quantum groups [RT][Tu]. We will see that the two seemingly exotic MTCs cannot be

constructed by using G finite or G a torus. Therefore, we will study whether or not they can be obtained from categories constructed from quantum groups.

Quantum groups are deformations of semi-simple Lie algebras. The standard quantum group theory does not have a well established theory to cover non-connected Lie groups. So our translation of Conjecture CSW to quantum group setting is not faithful since we will only study MTCs from deforming semi-simple Lie algebras. MTCs constructed in this way will be called **quantum group categories** in this paper, which are constructed mathematically (see [Tu] [BK] and the references therein). To remedy the situation to some extent, we will consider coset and orbifold constructions from quantum group categories in Section 7. It is known that coset and orbifold theories are included in the CSW theories by using appropriate compact Lie groups, in particular non-connected Lie groups [MS1].

There are new methods to construct MTCs. In particular many examples are constructed through VOAs and von Neumann algebras. Several experts in the mathematical community believe that those examples contain new MTCs that are not CSW MTCs. But as alluded above to prove such a statement is mathematically difficult. First even restricted to quantum group categories, the mathematical characterization of all MTCs from quantum group categories plus coset, orbifold, and simple current extension is hard, if not impossible. Secondly, the potentially new examples of MTCs are complicated measured by the number of simple object types. Another construction of MTCs is the quantum double, which is a categorical generalization of the Drinfeld double of quantum groups. Such MTCs give rise to TQFTs of the Turaev-Viro type [TV] and naturally arise in subfactor theory by A. Ocneanu's asymptotic inclusions construction (see [EK]). The categorical formulation of Ocneanu's construction is M. Müger's beautiful theorem that the quantum double of any spherical category is an MTC [M1]. The authors do not know how to construct quantum double TQFTs from CSW theory in general, except for the finite group case; hence general quantum double TQFTs are potentially exotic, and might be the only exotic ones. Maybe quantum double TQFTs can be constructed as CSW theory for some appropriate super-groups as Witten conjectured, but we are not aware of such mathematical theories.

The most famous examples of double TQFTs are related to the Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$. The Ocneanu construction gives rise to a unitary MTC which is the quantum double of a spherical category of 10 simple object types. This spherical category of 10 simple object types is not braided because there are simple objects X, Y such that $X \otimes Y$ is not isomorphic to $Y \otimes X$. Since the Haagerup subfactor cannot be constructed from quantum groups [AH], it is unlikely that the corresponding MTC is isomorphic to an MTC from quantum groups or a quantum double of a quantum group category. Recall that the standard construction of quantum group categories are always braided. But this MTC is difficult to study explicitly as the number of the simple object types, called the *rank* of an MTC,

is big. An alternative is to study the double of the even sectors of the Haagerup subfactor. The even sectors form a non-braided spherical category with 6 simple types; its quantum double is a unitary MTC of rank 12, which will be called **the Haagerup MTC**. There is another simpler category which has similar exoticness: the quantum double of $\frac{1}{2}E_6$, which is of rank 10. But note that a quantum double of a non-braided spherical category can be constructed by CSW theory sometimes. For example, if we double the group category S_3 of rank 6, which is not braided, we get a unitary MTC of rank 8, which is a Dijkgraaf-Witten MTC.

The spherical category $\frac{1}{2}E_6$ was brought to our attention by V. Ostrik [O1], and its double is worked out in [BEK] [Iz]. In V. Ostrik's paper on the classification of rank=3 fusion categories with braidings, he conjectured that there is only one set of fusion rules of rank=3 without braidings. This set of fusion rules is realized by the $\frac{1}{2}E_6$ fusion rules, and is known to be non-braided. If we denote by $1, x, y$ three representatives of the simple object types, their fusion rules are: $x^2 = 1 + 2x + y, xy = yx = x, y^2 = 1$. Note that we simply write tensor product as multiplication and will denote the $\frac{1}{2}E_6$ category as \mathcal{E} , and its double $\mathcal{Z}(\mathcal{E})$. If we denote by $1, \alpha, \alpha^*, \rho, {}_\alpha\rho, \alpha^*\rho$ six representatives of the even sectors of the Haagerup subfactor, their fusion rules are $\alpha\alpha^* = 1, \alpha^2 = \alpha^*, \alpha\rho = {}_\alpha\rho, \alpha^*\rho = \alpha^*\rho, \alpha\rho = \rho\alpha^*, \rho^2 = 1 + \rho + {}_\alpha\rho + \alpha^*\rho$. We will denote this rank 6 unitary fusion category by \mathcal{H} , and its double $\mathcal{Z}(\mathcal{H})$, the Haagerup MTC.

Our main result is:

Theorem 1.1. *Let \mathcal{E}, \mathcal{H} be the non-braided unitary spherical categories above, and $\mathcal{Z}(\mathcal{E}), \mathcal{Z}(\mathcal{H})$ be their quantum doubled MTCs. Then $\mathcal{Z}(\mathcal{E}), \mathcal{Z}(\mathcal{H})$*

- (1) *are prime, i.e. there are no non-trivial modular subcategories; hence are not a product of two MTCs;*
- (2) *have non-integral global quantum dimension D^2 , hence are not CSW MTCs for finite or torus Lie group G ;*
- (3) *are not quantum group categories;*
- (4) *are not quantum doubles of any braided fusion categories;*
- (5) *give rise to representations of $SL(2, \mathbb{Z})$ that factor over a finite group.*
- (6) *$\mathcal{Z}(\mathcal{E})$ gives rise to representations of the braid groups with infinite images;*
- (7) *$\mathcal{Z}(\mathcal{E})$ is a decomposable bimodule category over a pre-modular subcategory of quantum group type, but $\mathcal{Z}(\mathcal{H})$ has no such decompositions.*

It is known that all quantum double MTCs are non-chiral in the sense their topological central charges are 0, hence anomaly free in the sense that the representations afforded with all mapping class groups are linear representations rather than projective ones. This is a subtle point since the topological central charge is 0 does not imply the chiral central charge of the corresponding CFT, if there is one, is 0. The topological central charge is only defined modulo 8, so topological central charge being 0 means the chiral central charge of the corresponding CFT, if it exists, is 0 mod 8. It is possible that $\mathcal{Z}(\mathcal{E})$ or $\mathcal{Z}(\mathcal{H})$ can be constructed as an

orbifold of a quantum group category with topological central charge $= 0 \bmod 8$ or as a coset category of quantum group category. But as we will see in Section 7, an MTC of an orbifold CFT has global quantum dimension at least 4 times of that of the original MTC, so it is unlikely for either to be an orbifold. The coset construction is more complicated, but the constraint of central charge being multiples of 8 restricts the possible cosets significantly. We will leave a detailed analysis to the future.

To prove that any of the two MTCs are indeed not CSW MTCs for some pair (G, λ) , we need a classification of all CSW CFTs of central charge $0 \bmod 8$ and 10 or 12 primary fields. If the classification is simple enough, we may just examine the list to show that our exotic examples are not among the associated MTCs. This seems to be difficult.

The existence of a pre-modular subcategory in $\mathcal{Z}(\mathcal{E})$ raises an interesting possibility. The tensor sub-category generated by X_4 has 6 simple objects:

$$\{\mathbf{1}, Y, X_4, X_5, U, V\}.$$

The Bratteli diagram for decomposing tensor powers of X_4 is identical to that of a pre-modular category associated with the subcategory of non-spin representations of quantum \mathfrak{so}_3 at a 12th root of unity (see Prop. 2.1). This suggests the possibility that $\mathcal{Z}(\mathcal{E})$ might be related to an $O(3)$ -CSW MTC. Recall that $H^4(BO(3); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$, so for each level k , there are two CSW MTCs. It will be interesting to compare the 12th root of unity $O(3)$ -CSW theory for the nontrivial \mathbb{Z}_2 level with $\mathcal{Z}(\mathcal{E})$. A construction in CFT that we will not consider is the simple current extension. It is possible that $\mathcal{Z}(\mathcal{E})$ is a simple current extension of a quantum group category, or a sub-category of the simple extension of a quantum group category. On the other hand, $\mathcal{Z}(\mathcal{H})$ has no sub tensor categories, so the above discussion seems not applicable to $\mathcal{Z}(\mathcal{H})$. It is still possible that $\mathcal{Z}(\mathcal{E})$ or $\mathcal{Z}(\mathcal{H})$ can be constructed as the quantum double of a spherical quantum group category which is not braided. As far as we know there are no systematic ways to produce spherical quantum group categories that are not ribbon. But every fusion category comes from a *weak Hopf algebra*, therefore there will be no exotic MTCs if the term *quantum group* is too liberal [O2].

A sequence of potentially new chiral CFTs were recently constructed in [KL][Xu1]. The associated MTCs might be exotic. The simplest one in this sequence has an MTC equivalent to the mirror MTC of $SU(5)_1 \times SO(7)_1$ with chiral central charge 16.5. It will be interesting to analyze these categories.

The paper is organized as follows. In Section 2, we calculate the S, T matrices, and fusion rules of both TQFTs. Then we deduce several observations including (7) of Theorem 1.1. In Section 3, we prove (3) of Theorem 1.1 and neither $\mathcal{Z}(\mathcal{E})$ nor $\mathcal{Z}(\mathcal{H})$ is a product of two MTCs. Then both MTCs are prime because if there were a nontrivial modular subcategory of $\mathcal{Z}(\mathcal{E})$ or $\mathcal{Z}(\mathcal{H})$, [M2][Theorem 4.2] implies that $\mathcal{Z}(\mathcal{E})$ would be a non-trivial product of two MTCs, a contradiction. Sections

4, 5, 6 are devoted to the proofs of (4) (5) (6) of Theorem 1.1. In Section 7, we give evidence that neither theory is an orbifold or coset. In the appendix, we give an explicit description of the category $\mathcal{Z}(\mathcal{E})$ from the definition of half braidings.

As a final remark, regardless of the relevance to the Conjecture CSW, our work seems to be the most detailed study of non-quantum group TQFTs besides the Dijkgraaf-Witten, abelian and their direct product TQFTs. We also understand that some of the results are well-known to some experts, but are not well-documented.

2. CATEGORIES $\mathcal{Z}(\mathcal{E})$ AND $\mathcal{Z}(\mathcal{H})$

2.1. $\mathcal{Z}(\mathcal{E})$. The spherical category \mathcal{E} is studied in [BEK] [Iz], and the associated Turaev-Viro invariant is studied in [SW]. All spherical categories with the same set of fusion rules are worked out in great detail in [HH]. Those categories have three isomorphism classes of simple objects denoted by $1, x, y$, their fusion rules are: $x^2 = 1 + 2x + y, xy = yx = x, y^2 = 1$. The categories are called $\frac{1}{2}E_6$ because the fusion rules can be encoded by half of the Dynkin diagram E_6 . There is an essentially unique unitary spherical category with this set of fusion rules up to complex conjugation. We pick the same one as in [HH] as our \mathcal{E} , and all conceptual conclusions will be same for other choices except when specific complex parameters are involved. By direct computation from the definition (details are given in the appendix), we find that the quantum double of \mathcal{E} has 10 simple object types, of which 8 are self-dual and the other two are dual to each other. We label the 10 simple objects by $\mathbf{1} := (1, e_1), Y := (y, e_y), X_i := (x, e_{x_i})$ for $i = 1, 2, \dots, 5, U := (1 + x, e_{1+x}), V := (y + x, e_{y+x}),$ and $W := (1 + y + x, e_{1+y+x}),$ where the half-braiding notations as in the appendix are used here.

2.1.1. *S-matrix, T-matrix.* Once we have the list of all simple objects as half braidings as in the appendix, it is easy to compute the S -matrix, and the T -matrix. (The S, T matrices can also be computed from [Iz] and is also contained in [EP1].) In an MTC, we have $D = \sqrt{\sum_i d_i^2}$, where i goes over all simple object types. There are various names in the literature for D and D^2 . We call D^2 the global quantum dimension, and D the total quantum order. Total quantum order is not a standard terminology, and is inspired by the role that D plays in topological entropy for topological phases of matter. The S -matrix is $S = \frac{1}{D} \tilde{S}$, where the total quantum order $D = \sqrt{\dim(\mathcal{Z}(\mathcal{E}))} = 6 + 2\sqrt{3}$, and \tilde{S} is as follows:

$$\begin{pmatrix} 1 & 1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+2 & \sqrt{3}+2 & \sqrt{3}+3 \\ 1 & 1 & -\sqrt{3}-1 & -\sqrt{3}-1 & -\sqrt{3}-1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+2 & \sqrt{3}+2 & -\sqrt{3}-3 \\ \sqrt{3}+1 & -\sqrt{3}-1 & 0 & 0 & 0 & 2(\sqrt{3}+1) & -2(\sqrt{3}+1) & -\sqrt{3}-1 & \sqrt{3}+1 & 0 \\ \sqrt{3}+1 & -\sqrt{3}-1 & 0 & -i(\sqrt{3}+3) & i(\sqrt{3}+3) & -\sqrt{3}-1 & \sqrt{3}+1 & -\sqrt{3}-1 & \sqrt{3}+1 & 0 \\ \sqrt{3}+1 & -\sqrt{3}-1 & 0 & i(\sqrt{3}+3) & -i(\sqrt{3}+3) & -\sqrt{3}-1 & \sqrt{3}+1 & -\sqrt{3}-1 & \sqrt{3}+1 & 0 \\ \sqrt{3}+1 & \sqrt{3}+1 & 2(\sqrt{3}+1) & -\sqrt{3}-1 & -\sqrt{3}-1 & \sqrt{3}+1 & \sqrt{3}+1 & -\sqrt{3}-1 & -\sqrt{3}-1 & 0 \\ \sqrt{3}+1 & \sqrt{3}+1 & -2(\sqrt{3}+1) & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & -\sqrt{3}-1 & -\sqrt{3}-1 & 0 \\ \sqrt{3}+2 & \sqrt{3}+2 & -\sqrt{3}-1 & -\sqrt{3}-1 & -\sqrt{3}-1 & -\sqrt{3}-1 & -\sqrt{3}-1 & 1 & 1 & \sqrt{3}+3 \\ \sqrt{3}+2 & \sqrt{3}+2 & \sqrt{3}+1 & \sqrt{3}+1 & \sqrt{3}+1 & -\sqrt{3}-1 & -\sqrt{3}-1 & 1 & 1 & -\sqrt{3}-3 \\ \sqrt{3}+3 & -\sqrt{3}-3 & 0 & 0 & 0 & 0 & 0 & \sqrt{3}+3 & -\sqrt{3}-3 & 0 \end{pmatrix}$$

The T -matrix is diagonal with diagonal entries $\theta_{x_i}, i \in \Gamma$. The following are the entries:

$\theta_1 = 1, \theta_Y = -1, \theta_{X_1} = -i, \theta_{X_2} = \theta_{X_3} = e^{5\pi i/6}, \theta_{X_4} = e^{\pi i/3}, \theta_{X_5} = e^{-2\pi i/3}, \theta_U = 1, \theta_V = -1$, and $\theta_W = 1$.

Quantum dimensions of simple objects are among $\{1, 1 + \sqrt{3}, 2 + \sqrt{3}, 3 + \sqrt{3}\}$. Twists of simple objects are all 12th root of unity. Notice that every simple object is self-dual except that X_2 and X_3 which are dual to each other.

2.1.2. Fusion rules. The fusion rules for $\mathcal{Z}(\mathcal{E})$ can be obtained from the S -matrix via the Verlinde formula or more directly from the half-braidings. Set $F = \{1, Y, X_4, X_5, U, V\}$ and $M = \{X_1, X_2, X_3, W\}$. We record the non-trivial rules in the following:

$F \otimes F$	Y	X_4	X_5	U	V
Y	1	X_5	X_4	V	U
X_4	X_5	1 + X_4 + V	Y + X_5 + U	X_5 + U + V	X_4 + U + V
X_5	X_4	Y + X_5 + U	1 + X_4 + V	X_4 + U + V	X_5 + U + V
U	V	X_5 + U + V	X_4 + U + V	1 + X_4 + X_5 + U + V	Y + X_4 + X_5 + U + V
V	U	X_4 + U + V	X_5 + U + V	Y + X_4 + X_5 + U + V	1 + X_4 + X_5 + U + V

$F \otimes M$	X_1	X_2	X_3	W
Y	X_1	X_3	X_2	W
X_4	X_1 + W	X_3 + W	X_2 + W	X_1 + X_2 + X_3 + W
X_5	X_1 + W	X_2 + W	X_3 + W	X_1 + X_2 + X_3 + W
U	X_2 + X_3 + W	X_1 + X_3 + W	X_1 + X_2 + W	X_1 + X_2 + X_3 + $2W$
V	X_2 + X_3 + W	X_1 + X_3 + W	X_1 + X_3 + W	X_1 + X_2 + X_3 + $2W$

$M \otimes M$	X_1	X_2	X_3	W
X_1	1 + Y + X_4 + X_5	U + V	U + V	X_4 + X_5 + U + V
X_2	U + V	Y + X_4 + U	1 + X_5 + V	X_4 + X_5 + U + V
X_3	U + V	1 + X_5 + V	Y + X_4 + U	X_4 + X_5 + U + V
W	X_4 + X_5 + U + V	X_4 + X_5 + U + V	X_4 + X_5 + U + V	1 + Y + X_4 + X_5 + $2U$ + $2V$

From the fusion rules, we observe the following:

Proposition 2.1. $\mathcal{Z}(\mathcal{E})$ is a decomposable bimodule category over a pre-modular subcategory of quantum group type.

Proof. Observe that the tensor subcategory \mathcal{F} generated by X_4 has 6 simple objects; namely the simple objects in the set F above. The Bratteli diagram for decomposing tensor powers of X_4 is identical to that of a pre-modular category

associated with the subcategory of non-spin representations of quantum \mathfrak{so}_3 at a 12th root of unity. In fact, the eigenvalues of the braiding morphism c_{X_4, X_4} are identical to those of the fusion categories corresponding to BMW-algebras $BMW_n(q^2, q)$ with $q = e^{\pi i/6}$, and so it follows from the Tuba-Wenzl classification [TW] that these two categories are braided equivalent. Moreover, if one takes the semisimple abelian subcategory \mathcal{M} generated by the simple objects in the set M , one sees that $\mathcal{Z}(\mathcal{E}) = \mathcal{F} \oplus \mathcal{M}$ is \mathbb{Z}_2 -graded with $\mathcal{F} = \mathcal{Z}(\mathcal{E})_1$ and $\mathcal{M} = \mathcal{Z}(\mathcal{E})_{-1}$ and, moreover, \mathcal{M} is a bimodule category over \mathcal{F} . \square

This situation also has interesting connections to Conjecture 5.2 in Müger's [M2]. There he observes (Theorem 3.2) that if a modular category \mathcal{B} contains a semisimple tensor sub-category \mathcal{K} then $\dim(\mathcal{K}) \cdot \dim(C_{\mathcal{B}}(\mathcal{K})) = \dim(\mathcal{B})$, where $C_{\mathcal{B}}(\mathcal{K})$ is the centralizer subcategory of \mathcal{K} in \mathcal{B} . In the case of $\mathcal{F} \subset \mathcal{Z}(\mathcal{E})$ above, $C_{\mathcal{Z}(\mathcal{E})}(\mathcal{F})$ is the subcategory with simple objects $\mathbf{1}$ and Y , so that $\dim(C_{\mathcal{Z}(\mathcal{E})}(\mathcal{F})) = 2$. Müger calls this a *minimal modular extension* of \mathcal{F} . He conjectures that any unitary premodular category \mathcal{K} has a minimal modular extension, that is $\mathcal{K} \subset \hat{\mathcal{K}}$ where $\hat{\mathcal{K}}$ is modular and $\dim(\hat{\mathcal{K}}) = \dim(\mathcal{K}) \cdot \dim(C_{\hat{\mathcal{K}}}(\mathcal{K}))$. Notice also that \mathcal{F} above has at least two such: $\mathcal{F} \subset \mathcal{Z}(\mathcal{E})$ and $\mathcal{F} \subset \mathcal{C}(\mathfrak{sl}_2, e^{\pi i/12}, 12)$. This illustrates that the minimal modular extension fails to be unique, and in fact two such extensions can have different ranks!

2.2. $\mathcal{Z}(\mathcal{H})$. The modular category $\mathcal{Z}(\mathcal{H})$ has rank 12; we label and order the simple objects as follows: $\{\mathbf{1}, \pi_1, \pi_2, \sigma_1, \sigma_2, \sigma_3, \mu_1, \dots, \mu_6\}$ using an abbreviated version of the labeling found in [Iz]. The quantum dimensions of the non-trivial simple objects are $3d, 3d + 1$, and $3d + 2$ where $d = \frac{3+\sqrt{13}}{2}$, and the global quantum dimension is $\dim(\mathcal{Z}(\mathcal{H})) = (\frac{39+3\sqrt{13}}{2})^2$.

2.2.1. S -matrix, T -matrix. The modular S, T matrices are also contained in [EP2]. The review article [E] mentioned a paper in preparation that contains the explicit expressions of the x_i 's below. The total quantum order $D = \sqrt{\dim(\mathcal{Z}(\mathcal{H}))} = \frac{39+3\sqrt{13}}{2}$, and let x_i denote the six roots of the polynomial $x^6 - x^5 - 5x^4 + 4x^3 + 6x^2 - 3x - 1$ ordered as follows:

$$\begin{aligned} x_1 &\approx 0.7092097741, \quad x_2 \approx 1.497021496, \quad x_3 \approx 1.941883635, \\ x_4 &\approx -0.2410733605, \quad x_5 \approx -1.136129493, \quad \text{and} \quad x_6 \approx -1.770912051. \end{aligned}$$

With this notation, the S -matrix for $\mathcal{Z}(\mathcal{H})$ is $S = \tilde{S}/D$, where:

$$\tilde{S} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 3d+1 & 3d+2 & 3d+2 & 3d+2 & 3d+2 \\ 3d+1 & 1 & 3d+2 & 3d+2 & 3d+2 & 3d+2 \\ 3d+2 & 3d+2 & 6d+4 & -3d-2 & -3d-2 & -3d-2 \\ 3d+2 & 3d+2 & -3d-2 & 6d+4 & -3d-2 & -3d-2 \\ 3d+2 & 3d+2 & -3d-2 & -3d-2 & -3d-2 & 6d+4 \\ 3d+2 & 3d+2 & -3d-2 & -3d-2 & 6d+4 & -3d-2 \end{pmatrix},$$

$$B = 3d \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, C = 3d \begin{pmatrix} x_1 & x_3 & x_6 & x_2 & x_4 & x_5 \\ x_3 & x_1 & x_2 & x_6 & x_5 & x_4 \\ x_6 & x_2 & x_4 & x_5 & x_1 & x_3 \\ x_2 & x_6 & x_5 & x_4 & x_3 & x_1 \\ x_4 & x_5 & x_1 & x_3 & x_6 & x_2 \\ x_5 & x_4 & x_3 & x_1 & x_2 & x_6 \end{pmatrix}$$

Now fix $\gamma = e^{2\pi i/13}$. The T-matrix is given in [Iz] and is the diagonal matrix with diagonal entries

$$(1, 1, 1, 1, e^{2\pi i/3}, e^{-2\pi i/3}, \gamma^2, \overline{\gamma}^2, \gamma^5, \overline{\gamma}^5, \gamma^6, \overline{\gamma}^6),$$

2.2.2. *Fusion rules.* The fusion rules are obtained from the S matrix via the Verlinde formula. The fusion matrices for the objects π_1 , π_2 and μ_1 are:

[illegible]

$$N_{\pi_2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$N_{\mu_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The remaining fusion matrices can be obtained from these three by permuting the rows and columns. The specific permutation that effects these similarity transformations are deduced by comparing the rows of the S matrix above. For example, N_{σ_i} is obtained from N_{π_2} by the transposition $\pi_2 \leftrightarrow \sigma_i$, since this transposition transforms the row of the S matrix labeled by π_2 with the row labeled by σ_i . Similarly, N_{μ_2} can be obtained from N_{μ_1} by a permutation of rows and columns, specifically, $\mu_1 \leftrightarrow \mu_2$, $\mu_3 \leftrightarrow \mu_4$ and $\mu_5 \leftrightarrow \mu_6$ converts N_{μ_1} to N_{μ_2} . The required permutation is not always order 2, for example, N_{μ_3} is obtained from N_{μ_1} via the permutation (written in cycle notation) $(\mu_1, \mu_3, \mu_5)(\mu_2, \mu_4, \mu_6)$.

From these fusion rules we obtain the following:

Proposition 2.2. *The modular category $\mathcal{Z}(\mathcal{H})$ has no non-trivial tensor subcategories.*

Proof. First observe that $\pi_2 \otimes \pi_2$ contains every simple object. Since the diagonal entries of N_{π_2} are all positive except for the trivial object, we conclude that π_2

appears in $X \otimes X$ for *every* non-trivial simple object in $\mathcal{Z}(\mathcal{H})$ since every object is self-dual. Thus the tensor subcategory generated by *any* non-trivial simple object is all of $\mathcal{Z}(\mathcal{H})$. \square

3. MTCs FROM QUANTUM GROUPS

From any simple Lie algebra \mathfrak{g} and $q \in \mathbb{C}$ with q^2 a primitive ℓ th root of unity one may construct a ribbon category $\mathcal{C}(\mathfrak{g}, q, \ell)$ (see *e.g.* [BK]). One can also construct such categories from semisimple \mathfrak{g} , but the resulting category is easily seen to be a direct product of those constructed from simple \mathfrak{g} . We shall say these categories (or direct products of them) are of **quantum group type**. There is an (often overlooked) subtlety concerning the degree ℓ of q^2 and the unitarizability of the category $\mathcal{C}(\mathfrak{g}, q, \ell)$. Let m denote the maximal number of edges between any two nodes of the Dynkin diagram for \mathfrak{g} with \mathfrak{g} simple, so that $m = 1$ for Lie types ADE , $m = 2$ for Lie types BCF_4 and $m = 3$ for Lie type G_2 . Provided $m \mid \ell$, $\mathcal{C}(\mathfrak{g}, q, \ell)$ is a unitary category for $q = e^{\pi i/\ell}$ (see [We]). If $m \nmid \ell$, this is not always true and in fact there is usually no choice of q to make $\mathcal{C}(\mathfrak{g}, q, \ell)$ unitary (see [R2] and [R3]). In [Fin] it is shown that the fusion category associated with level k representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is tensor equivalent to $\mathcal{C}(\mathfrak{g}, q, \ell)$ for $\ell = m(k + h_{\mathfrak{g}})$ where $h_{\mathfrak{g}}$ is the dual Coxeter number. In these cases the categories are often denoted (X_r, k) , where \mathfrak{g} is of Lie type X with rank r and $k = \ell/m - h_{\mathfrak{g}}$ is the level. We will use this abbreviated notation except when $m \nmid \ell$.

Our goal is to prove the following:

Theorem 3.1. *The modular categories $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ are not monoidally equivalent to any category of quantum group type.*

Before we proceed to the proof, we give a few more details on the categories $\mathcal{C}(\mathfrak{g}, q, \ell)$ with \mathfrak{g} simple. The (isomorphism classes of) objects in $\mathcal{C}(\mathfrak{g}, q, \ell)$ are labeled by a certain finite subset C_{ℓ} of the dominant weights of \mathfrak{g} . The size $|C_{\ell}|$ is the *rank* of $\mathcal{C}(\mathfrak{g}, q, \ell)$; for \mathfrak{g} simple, generating functions for $|C_{\ell}|$ are found in [R1]. For any object X , we have that $X \cong X^*$ if and only if the corresponding simple \mathfrak{g} -module V satisfies $V \cong V^*$, in which case we say that X is *self-dual*. If every object is self-dual, we will say the category itself is self-dual. A simple object X_{λ} is self-dual if and only if $-w_0(\lambda) = \lambda$, where w_0 is the longest element of the Weyl group of \mathfrak{g} , (see *e.g.* [GW][Exercise 5.1.8.4]). In a ribbon category (such as $\mathcal{C}(\mathfrak{g}, q, \ell)$) it is always true that $X^{**} \cong X$ for any object X , so non-self dual objects always appear in pairs. Observe also that the unit object $\mathbf{1}$ is always self-dual. The twists for the simple objects in $\mathcal{C}(\mathfrak{g}, q, \ell)$ are powers of $q^{1/M}$ where M is the order of the quotient group P/Q of the weight lattice P by the root lattice Q . In particular, $M = (r + 1)$ for A_r , and for all other Lie types $M \leq 4$.

Proof. Observe that $\mathcal{Z}(\mathcal{E})$ has rank 10 and has exactly one pair of simple non-self-dual objects, and 8 simple self-dual objects (up to isomorphism). For $\mathcal{Z}(\mathcal{E})$

the statement follows from the following fact (to be established below): *no rank 10 category of quantum group type has exactly one pair of simple non-self-dual objects.*

It is immediate that $\mathcal{Z}(\mathcal{E})$ cannot be a direct product $\mathcal{F} \boxtimes \mathcal{T}$ of two non-trivial modular categories. First notice that one may assume that $\text{rank}(\mathcal{F}) = 2$ and $\text{rank}(\mathcal{T}) = 5$. Since rank 2 modular category are all self-dual and simple non-self-dual objects appear in pairs, $\mathcal{F} \boxtimes \mathcal{T}$ has either 0, 4 or 8 simple non-self-dual objects, while $\mathcal{Z}(\mathcal{E})$ has exactly 2.

Next we observe that if \mathfrak{g} is of Lie type $A_1, B_r, C_r, D_{2t}, E_7, E_8, F_4$ or G_2 , all of the simple objects in the corresponding category are self-dual, since -1 is the longest element of the Weyl group. So we may immediately eliminate categories of these Lie types from consideration. This leaves only Lie types A_r ($r \geq 2$), D_r ($r \geq 5$ and odd) and E_6 as possibilities.

From [R1] we have the following generating functions for $|C_\ell|$, the rank of $\mathcal{C}(\mathfrak{g}, q, \ell)$:

- (1) A_r : $\frac{1}{(1-x)^{r+1}} = \sum_{k=0}^{\infty} \binom{r+k}{k} x^k$.
- (2) D_r : $\frac{1}{(1-x)^4(1-x^2)^{r-3}} = 1 + 4x + (r+7)x^2 + (8+4r)x^3 + \dots$
- (3) E_6 : $\frac{1}{(1-x)^3(1-x^2)^3(1-x^3)} = 1 + 3x + 9x^2 + 20x^3 + 42x^4 + \dots$

where the coefficient of x^k is the rank of $\mathcal{C}(\mathfrak{g}, q, \ell)$, where $\ell = k + h$ with h the dual Coxeter number of the root system of \mathfrak{g} .

To determine the rank 10 type A categories, we must solve $\binom{r+k}{k} = 10$ for (r, k) . The only positive integer solutions are $(1, 9)$, $(9, 1)$, $(3, 2)$ and $(2, 3)$. These correspond to $(A_1, 9)$ and $(A_9, 1)$ (i.e. at 11th roots of unity) and $(A_2, 3)$ and $(A_3, 2)$ (i.e. at 6th roots of unity). The category $(A_1, 9)$ has only self-dual objects, and the remaining three categories each have at least 4 non-self-dual objects. For type D_r with $r \geq 5$ and E_6 we see that no rank 10 categories appear.

Next let us consider the (self-dual) rank 12 category $\mathcal{Z}(\mathcal{H})$. Since $\mathcal{Z}(\mathcal{H})$ has no tensor subcategories by Prop. 2.2, $\mathcal{Z}(\mathcal{H})$ is not the product of two modular categories. Using the generating functions from [R1] we determine all rank 12 self-dual quantum group categories. The following pairs (X_r, k) are the rank r Lie type X quantum groups at level k that have exactly 12 simple objects:

$$\{(G_2, 5), (A_1, 11), (B_8, 2), (C_{11}, 1), (D_5, 2), (E_7, 3)\}.$$

In addition the categories $\mathcal{C}(\mathfrak{so}_5, q, 9)$ and $\mathcal{C}(\mathfrak{so}_{11}, q, 13)$ have rank 12. The twists θ_i in $\mathcal{Z}(\mathcal{H})$ include 13th roots of unity, so that we may immediately eliminate all of the above except $(A_1, 11)$, $(C_{11}, 1)$ and $\mathcal{C}(\mathfrak{so}_{11}, q, 13)$. By a level-rank duality theorem in [R2] the pairs $(A_1, 11)$ and $\mathcal{C}(\mathfrak{so}_{11}, q, 13)$ have the same fusion rules, and moreover each contains a tensor subcategory eliminating them from consideration. To eliminate $(C_{11}, 1)$ we must work a little harder. The category $\mathcal{C}(\mathfrak{sp}_{22}, e^{\pi i/26}, 26)$ contains a simple object $X := X_{(1,0,\dots,0)}$ corresponding to the 22 dimensional

representation of \mathfrak{sp}_{22} . The quantum dimension of this object is

$$\frac{[11][24]}{[1][12]} = 2 \cos(\pi/13) = 1.94188 \dots,$$

where $[n]$ is the usual q -number at $q = e^{\pi i/26}$. Since the simple objects in $\mathcal{Z}(\mathcal{H})$ have quantum dimensions in $\{1, 3d, 3d+1, 3d+2\}$, where $d = \frac{3+\sqrt{13}}{2} > 3$, we see that $\mathcal{Z}(\mathcal{H})$ cannot be obtained from $(C_{11}, 1)$. Thus $\mathcal{Z}(\mathcal{H})$ is not a quantum group type category. \square

4. DOUBLED CATEGORIES

Another way in which modular categories may be constructed is as the quantum double $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category [M1]. In this section we will prove the following:

Theorem 4.1. *The modular categories $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ are not braided monoidally equivalent to the double of any braided fusion category.*

Proof. First observe that by [K][Corollary XIII.4.4] and [M1][Lemma 7.1] the double $\mathcal{Z}(\mathcal{C})$ of a braided fusion category \mathcal{C} contains a braided tensor subcategory equivalent to \mathcal{C} . In the case of $\mathcal{Z}(\mathcal{H})$, we showed in Prop. 2.2 that the only tensor subcategories are the trivial subcategory and $\mathcal{Z}(\mathcal{H})$ itself. So $\mathcal{Z}(\mathcal{H})$ is not the double of *any* braided fusion category.

Suppose that $\mathcal{Z}(\mathcal{E})$ is braided monoidally equivalent to the double $\mathcal{Z}(\mathcal{C})$ of some braided fusion category \mathcal{C} , then \mathcal{C} is equivalent to some braided fusion subcategory $\mathcal{C}' \subset \mathcal{Z}(\mathcal{E})$. Since $\mathcal{Z}(\mathcal{E})$ is modular we may further assume that \mathcal{C}' is pre-modular and that $\dim(\mathcal{C}')^2 = \dim(\mathcal{Z}(\mathcal{E})) = (6+2\sqrt{3})^2$ by [M1][Theorem 1.2]. Thus $\dim(\mathcal{C}') = 6+2\sqrt{3}$ which implies that there exists some simple object $X \in \mathcal{C}'$ with $\dim(X) \neq 1$. Let X be such an object, then $\dim(X) \in \{1+\sqrt{3}, 2+\sqrt{3}, 3+\sqrt{3}\}$ since X would be a simple object of $\mathcal{Z}(\mathcal{E})$. The inequality $\dim(\mathbf{1})^2 + \dim(X)^2 \leq \dim(\mathcal{C}') = 6+2\sqrt{3}$ implies that $\dim(X) = 1+\sqrt{3}$, and this forces \mathcal{C}' to have 3 simple objects of dimension 1, 1 and $1+\sqrt{3}$. But it is known ([O1]) that no such category can be braided, therefore, $\mathcal{Z}(\mathcal{E})$ cannot be a double of any braided fusion category. \square

5. $SL(2, \mathbb{Z})$ IMAGE

Every TQFT gives rise to a projective representation of $SL(2, \mathbb{Z})$ via

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow T.$$

If the TQFT has a corresponding RCFT, then the resulting representation of $SL(2, \mathbb{Z})$ has a finite image group, and the kernel is a congruence subgroup [Ba][Xu3]. It is an open question if this is true for every TQFT. In particular, any TQFT

whose representation of $SL(2, \mathbb{Z})$ has an infinite image or has a non-congruence subgroup kernel is not a CSW TQFT. But representations of $SL(2, \mathbb{Z})$ from $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ both behave as those of TQFTs from RCFTs.

Theorem 5.1. (1) *The representation of $SL(2, \mathbb{Z})$ from $\mathcal{Z}(\mathcal{E})$ has a finite image in $U(10)$;*
 (2) *The representation of $SL(2, \mathbb{Z})$ from $\mathcal{Z}(\mathcal{H})$ has a finite image in $U(12)$, and its kernel is a congruence subgroup.*

Proof. First let us consider the category $\mathcal{Z}(\mathcal{E})$. We wish to show that the 10-dimensional unitary representation of $SL(2, \mathbb{Z})$ given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow T$$

has finite image, where S and T are as in Section 2.1.1. We accomplish this as follows: Let $G = \langle S, T \rangle$ be the group generated by S and T . Observe that we immediately have the following relations in G , since T is a diagonal matrix whose nonzero entries are 12th roots of unity:

$$(5.2) \quad S^4 = T^{12} = I, \quad (ST)^3 = S^2.$$

Additionally, we find the following relation:

$$(5.3) \quad (T^4 S T^6 S)^6 = I.$$

In fact, $A := (T^4 S T^6 S)^2$ is a diagonal order 3 matrix and will play an important role in what follows.

Now consider the normal closure, N , of the cyclic subgroup generated by A in G . We need to see how G acts on N , and for this it is enough to understand the action of S and T on a set of generators of N . We will employ the standard notation for conjugation in a group: $g^h := hgh^{-1}$. Defining $B := A^S$, $C := B^T$ and $D := C^S$, we find that a set of generators for N is $\{A, B, C, D\}$. This is established by determining the conjugation action of S and T on these generators as follows:

$$(5.4) \quad \begin{aligned} A^T &= A, & A^S &= B, & B^T &= C, & B^S &= A, \\ C^T &= D, & C^S &= D, & D^T &= B, & D^S &= C. \end{aligned}$$

Thus the subgroup generated by $\{A, B, C, D\}$ is normal and contains A , hence is equal to N . Furthermore, one has the following relations in N :

$$(5.5) \quad A^3 = I, \quad B^A = D, \quad C^A = B, \quad D^A = C, \quad D^B = A$$

which are sufficient to determine all other conjugation relations among the generators of N . Having established these relations in G , we proceed to analyze the structure of the abstract group \widehat{G} on generators $\{s, t, a, b, c, d\}$ satisfying relations (5.2), (5.3), (5.4) and (5.5) (where S is replaced by s etc.). Clearly G is a quotient of \widehat{G} since these relations hold in G .

First observe N is a quotient of the abstract group

$$\widehat{N} := \langle a, b, c, d \mid a^3 = I, b^a = d, c^a = b, d^a = c, d^b = a \rangle$$

and that $\widehat{N} \triangleleft \widehat{G}$. We compute $|\widehat{N}| = 24$, so that if $\widehat{H} := \widehat{G}/\widehat{N}$ is a finite group, then \widehat{G} is finite hence G is finite. Next observe that

$$\widehat{H} = \langle s, t \mid s^4 = t^{12} = (t^4 s t^6 s)^2 = I, (st)^3 = s^2 \rangle,$$

i.e. with relations (5.2) together with $(t^4 s t^6 s)^2 = I$, and we have a short exact sequence

$$1 \rightarrow \widehat{N} \rightarrow \widehat{G} \rightarrow \widehat{H} \rightarrow 1.$$

Using MAPLE, we find that $|\widehat{H}| = 1296$ so that $|\widehat{G}| = (24)(1296) = 31104$. Thus $|G|$ is a finite group of order dividing $31104 = (2^7)(3^5)$. Note that since $|G|$ is not divisible by 10 we can conclude that the representation above is reducible.

Incidentally, the group $\mathrm{SL}(2, \mathbb{Z}/36\mathbb{Z})$ (i.e. $\mathrm{SL}(2, \mathbb{Z})$ with entries taken modulo 36) has order 31104 just as \widehat{G} above does. While we expect that the kernel of $\mathrm{SL}(2, \mathbb{Z}) \rightarrow G$ is a congruence subgroup, it is *not* true that $\widehat{G} \cong \mathrm{SL}(2, \mathbb{Z}/36\mathbb{Z})$, according to computations with GAP ([GAP]). However, both of these groups are solvable, and their (normal) Sylow-2 and Sylow-3 subgroups *are* isomorphic. This implies that \widehat{G} and $\mathrm{SL}(2, \mathbb{Z}/36\mathbb{Z})$ are semi-direct products of the same two groups. This of course does not imply that the kernel of $\mathrm{SL}(2, \mathbb{Z}) \rightarrow G$ is non-congruence, merely that the obvious guess is incorrect.

For $\mathcal{Z}(\mathcal{H})$ we use the 12×12 matrices S and T found in Section 2.2.1. For N odd, the following generators and relations for $\mathrm{PSL}(2, \mathbb{Z}/N\mathbb{Z})$ are found in [Su]:

$$(AB)^3 = A^2 = B^N = (B^4 A B^{\frac{N+1}{2}} A)^2 = I.$$

Setting $A = S$, $B = T$ and $N = 39$, we easily verify these relations, so that the image of $\mathrm{SL}(2, \mathbb{Z})$ for the Haagerup MTC $\mathcal{Z}(\mathcal{H})$ is finite and the kernel is a congruence subgroup. □

6. REPRESENTATION OF THE BRAID GROUPS

Representations of the braid group \mathcal{B}_n can be obtained from any simple object X in a braided tensor category \mathcal{C} . The construction is as follows. For $1 \leq i \leq n-1$, Let β_i be the usual generators of the braid group satisfying $\beta_i \beta_{i+1} \beta_i = \beta_{i+1} \beta_i \beta_{i+1}$ and $\beta_i \beta_j = \beta_j \beta_i$ for $|i - j| > 1$. The braiding operator $c_{X,X} \in \mathrm{End}(X^{\otimes 2})$ acts on $\mathrm{End}(X^{\otimes 2})$ by composition, and we define invertible operators in $\mathrm{End}(X^{\otimes n})$ by: $\phi_X^n(\beta_i) = \mathrm{Id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes n-i-1}$. This defines a representation $\mathcal{B}_n \rightarrow \mathrm{GL}(\mathrm{End}(X^{\otimes n}))$ by $\beta_i \cdot f \rightarrow \phi_X^n(\beta_i) \circ f$ which is unitarizable in case \mathcal{C} is a unitary ribbon category.

These representations are rarely irreducible. In fact, if Y is a simple subobject of $X^{\otimes n}$, then $\mathrm{Hom}(Y, X^{\otimes n})$ is (isomorphic to) a \mathcal{B}_n -subrepresentation of $\mathrm{End}(X^{\otimes n})$,

since $\text{Hom}(Y, X^{\otimes n})$ is obviously stable under composition with $\phi_X^n(\beta_i)$. However, the \mathcal{B}_n -subrepresentations of the form $\text{Hom}(Y, X^{\otimes n})$ are not irreducible for all Y unless the algebra $\text{End}(X^{\otimes n})$ is generated by $\{\phi_X^n(\beta_i)\}$. It is a technically difficult problem to determine the irreducible constituents of $\text{End}(X^{\otimes n})$ as a \mathcal{B}_n -representation; few general techniques are available. One useful criterion is the following proposition [TW][Lemma 5.5]:

Proposition 6.1. *Suppose X is a simple self-dual object in a ribbon category \mathcal{C} , such that*

- (a) $X^{\otimes 2}$ decomposes as a direct sum of d distinct simple objects X_i and
- (b) $\phi_X^2(\beta_1)$ has d distinct eigenvalues.

Then \mathcal{B}_3 acts irreducibly on $\text{Hom}(X, X^{\otimes 3})$.

A generalization of this result to spaces of the form $\text{Hom}(Y, X^{\otimes 3})$ and for repeated eigenvalues would be of considerable value.

An important question for a given MTC \mathcal{C} is the following: do these representations of \mathcal{B}_n factor over finite groups for all X and all n , or is there a choice of X so that the image of \mathcal{B}_n is infinite (say, for all $n \geq 3$)?

Proposition 6.2. *Representations of the braid groups $\mathcal{B}_n, n \geq 3$ from the simple object X_4 of $\mathcal{Z}(\mathcal{E})$ has a dense image in the projective unitary group.*

Proof. For the simple object X_4 of $\mathcal{Z}(\mathcal{E})$, we have $X_4^{\otimes 2} = \mathbf{1} + X_4 + V$. So $\text{Hom}(\mathbf{1}, X_4^2), \text{Hom}(X_4, X_4^2), \text{Hom}(V, X_4^2)$ are all 1-dimensional. Set $q = e^{\pi i/6}$. The eigenvalues of braiding $\phi_{X_4}^2(\beta_i)$ are computed as: $a = \frac{1-\sqrt{3}i}{2} = q^{-2}, b = \frac{-\sqrt{3}+i}{2} = -q^{-1}, c = \frac{\sqrt{3}+i}{2} = q$. Since these eigenvalues are distinct, it follows from Prop. 6.1 that the representation is irreducible. The projective order of braid generators of \mathcal{B}_3 is 12 because 12 is the smallest m so that $a^m = b^m = c^m$. It follows from [LRW][Prop. 6.8] that the image of the representation of \mathcal{B}_n for $n \geq 3$ afforded by $\mathcal{Z}(\mathcal{E})$ with each braid strand colored by X_4 is infinite, and dense in the projective unitary group. \square

For the Haagerup MTC $\mathcal{Z}(\mathcal{H})$ we cannot conclude the image is infinite without considerably more work. The necessary techniques are somewhat ad hoc and go beyond the scope of this paper. We plan to give an account of these techniques in a subsequent article.

We give a brief explanation of the difficulties one encounters in this case. The smallest non-trivial representation of \mathcal{B}_3 is 7 dimensional, for example acting on the vector space $\text{Hom}(\mu_2, \mu_1^{\otimes 3})$. Set $\gamma = e^{2\pi i/13}$ as above. The eigenvalues of the operators $(\phi_{\mu_1}^3(\beta_i))^2$ restricted to this space are

$$\gamma^4 \cdot \{1, 1, e^{\pm 2\pi i/3}, \gamma^{\pm 2}, \gamma^{-5}\}$$

which can easily be computed from the twists. However, we do not know that this representation is irreducible, or indeed, not a sum of 1-dimensional representations,

since Prop. 6.1 only applies to spaces of the form $\text{Hom}(X, X^{\otimes 3})$. A 10 dimensional representation of \mathcal{B}_3 with the right form is $\text{Hom}(\mu_1, \mu_1^{\otimes 3})$, and the corresponding eigenvalues of $(\phi_{\mu_1}^3(\beta_i))^2$ are:

$$\gamma^4\{1, 1, 1, e^{\pm 2\pi i/3}, \gamma^{\pm 2}, \gamma^{-5}, \gamma^{\pm 6}\}.$$

But the eigenvalues of $\phi_{\mu_1}^3(\beta_i)$ are some choices of square roots of these values which will clearly not be distinct.

The technique for showing that the image is infinite is as follows. First find an irreducible subrepresentation of dimension d . Next verify that the corresponding image is not imprimitive by checking the “no-cycle condition” of [LRW] or by some other means. Then check that the projective order of the images of β_i does not occur for primitive linear groups of degree d by checking the lists in [Feit1] and [Feit2] of primitive linear groups of degree $d \leq 10$. For example, if \mathcal{B}_3 acts primitively on some d -dimensional subrepresentation W of $\text{Hom}(\mu_1, \mu_1^{\otimes 3})$ and has a) $2 \leq d$ and b) both 3rd and 13th roots of unity occur as eigenvalues of $(\phi_{\mu_1}^3(\beta_i))^2$ acting on W then the image must be infinite.

7. CENTRAL CHARGE AND ORBIFOLD/COSET CONSTRUCTIONS

Since we do not know how to cover general compact Lie groups G in the quantum group setting, we will restrict our discussion to the semi-simple cases. The orbifold and coset CFTs for WZW models with semi-simple Lie groups G have been constructed mathematically. Although complete analysis of all possible orbifold and coset candidates for $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ seems impossible, we will give evidence that orbifold and coset constructions are unlikely to realize them.

If a TQFT comes from a RCFT, then a relation between the topological central charge of the TQFT and the chiral central charge of the corresponding RCFT exists. The topological central charge of a TQFT is defined as follows:

Definition 7.1. *Let d_i be the quantum dimensions of all simple types $X_i, i = 1, \dots, n$ of an MTC \mathcal{C} , θ_i be the twists, define the total quantum order of \mathcal{C} to be $D = \sqrt{\sum_{i=1}^n d_i^2}$, and $D_+ = \sum_{i=1}^n \theta_i d_i^2$. Then $\frac{D_+}{D} = e^{\frac{\pi i c}{4}}$ for some rational number c . The rational number c defined modulo 8 will be called the topological central charge of the MTC \mathcal{C} .*

Each CSW TQFT corresponds to a RCFT. The chiral central charge of the RCFT is a rational number c_v . We have the following:

Proposition 7.2. *If a TQFT has a corresponding RCFT, then $c_v = c \bmod 8$, in particular this is true for CSW TQFTs.*

Proof. This relation first appeared in [Re]. For another explanation, see [Ki2] on Page 66. For general unitary TQFTs, it is not known if the boundary theories are always RCFTs, and it is an open question if there is a similar identity. See the references in [Ki2] and [Witt2].

□

Since $\mathcal{Z}(\mathcal{E}), \mathcal{Z}(\mathcal{H})$ have topological central charge $= 0$, a corresponding CFT, if exist, would have chiral central charge $= 0 \bmod 8$. To rule out the possibility of coset and orbifold constructions of $\mathcal{Z}(\mathcal{E})$ or $\mathcal{Z}(\mathcal{H})$, we need to have a list of all chiral central charge $= 0 \bmod 8$ CFTs, and their orbifolds. This question seems hard. So we will only consider, as examples, the case of CFTs with chiral central charge 0 or 24. Even with this restriction, the problem is still hard, so we will further restrict our discussion to simple quantum group categories, i.e. those from simple Lie algebras plus their orbifolds and certain cosets.

As shown in [M3], the orbifold construction in CFT cannot be formulated purely in a categorical way (coset construction has not been attempted systematically in the categorical framework). In the case of quantum group categories, this problem can be circumvented by the following detour: the corresponding CFTs are WZW models, coset and orbifold CFTs of WZW models are mathematically constructed (see [M3][Xu2] and the references therein). We will then take the corresponding MTCs of the resulting CFTs as the cosets or orbifolds of the quantum group categories.

We collect some facts about orbifold and coset CFTs that we need in this section from [M3][Xu2][DMS]. Given a simple Lie algebra \mathfrak{g} and a level k , the WZW CFT has chiral central charge $c = \frac{k \cdot \dim \mathfrak{g}}{k + h_{\mathfrak{g}}}$, where $h_{\mathfrak{g}}$ is the dual Coxeter number of \mathfrak{g} . Given a CFT with a discrete finite automorphism group G on its chiral algebra A , then the orbifold CFT based on $\text{Rep}(A^G)$ of the fixed algebra has the same chiral central charge.

The coset construction is complicated and is defined for any pair of Lie groups $H \subset G$. We will restrict ourselves to the cases that $\mathfrak{p} \subset \mathfrak{g}$ such that both are simple, and \mathfrak{p} is an isolated maximal subalgebra as in Tables 2 and 5 of [BB]. The total quantum order $D_{G/H}$ of a coset $H \subset G$ MTC is $D_{G/H} = \frac{D_G}{D_H} \cdot d^2(G/H)$, where $d^2(G/H)$ is the index of type II_1 subfactors [Xu2]. By Jones' celebrated theorem [Jo], if $d(G/H) \leq 2$, then $d(G/H) = 2\cos(\pi/r)$ for some $r \geq 3$.

Given a simple Lie algebra \mathfrak{g} , and a level k . Let \mathfrak{p} be a simple subalgebra, and χ be the Dynkin embedding index of \mathfrak{p} in \mathfrak{g} . Then the central charge of the resulting CFT $\mathfrak{g}/\mathfrak{p}$ is $c_{\mathfrak{g}/\mathfrak{p}} = \frac{k \cdot \dim \mathfrak{g}}{k + h_{\mathfrak{g}}} - \frac{\chi \cdot k \cdot \dim \mathfrak{p}}{\chi \cdot k + h_{\mathfrak{p}}}$. Recall the Dynkin embedding index for a pair of simple Lie algebras $\mathfrak{p} \subset \mathfrak{g}$: let λ be a highest weight of \mathfrak{g} , then $\lambda = \bigoplus_{\mu \in P_+} b_{\lambda\mu} \mu$, where P_+ is the set of dominant weights of \mathfrak{p} , then $\chi_{\mathfrak{g}/\mathfrak{p}} = \sum_{\mu \in P_+} b_{\lambda\mu} \frac{\chi_{\mu}}{\chi_{\lambda}}$, where $\chi_{\lambda} = \frac{\dim \lambda \cdot (\lambda, \lambda + 2\rho_{\mathfrak{g}})}{2 \cdot \dim \mathfrak{g}}$, $\chi_{\mu} = \frac{\dim \mu \cdot (\mu, \mu + 2\rho_{\mathfrak{p}})}{2 \cdot \dim \mathfrak{p}}$.

Proposition 7.3. (1) *If the total quantum order of a unitary MTC \mathcal{C} from a CFT is D , then any nontrivial orbifold of \mathcal{C} has total quantum order $\geq 2D$.*
 (2) *A chiral central charge 24 unitary CFT from a simple Lie algebra is one of the following:*

$$(A_6, 7), (A_{24}, 1), (B_{12}, 2), (C_4, 10), (D_{24}, 1).$$

Moreover, neither $\mathcal{Z}(\mathcal{E})$ nor $\mathcal{Z}(\mathcal{H})$ is an orbifold of those CFTs.

- (3) The only chiral central charge 24 unitary coset CFT of the form $\mathfrak{g}/\mathfrak{p}$ for simple $\mathfrak{p}, \mathfrak{g}$ in Tables 2 and 5 of [BB] is from the embedding $A_7 \subset D_{35}$ with embedding index $\chi = 10$, and D_{35} is at level $k = 2$. The resulting coset TQFT is neither $\mathcal{Z}(\mathcal{E})$ nor $\mathcal{Z}(\mathcal{H})$.

Proof. (1): Let D_A be the total quantum order of the MTC corresponds to CFT A , then the orbifold MTC has total quantum order $|G| \cdot D_A$, and the inequality follows.

(2): A chiral unitary central charge 0 CFT is trivial, and the orbifolds of the trivial CFT are (twisted) quantum double of finite groups whose quantum dimensions are all integers [BK]. But we know $\mathcal{Z}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{H})$ both have non-integral quantum dimensions, hence they are not orbifolds of the trivial CFT.

In [Sc], 71 CFTs of chiral central charge 24 are listed. A simple inspection gives our list for simple algebras. More directly, we can find the list by solving Diophantine equations $24 = \frac{k \cdot \dim \mathfrak{g}}{k + h_{\mathfrak{g}}}$ for all simple Lie algebras.

$(A_{24}, 1)$ corresponds to $SU(25)$ at a 26th root of unity. This is a rank 25 abelian theory, with all categorical dimensions of simple objects equal to 1. Similarly, corresponding to $(D_{24}, 1)$ is an abelian rank 4 category. So any orbifold theory will have global quantum dimension $N^2 \cdot 25$ or $N^2 \cdot 4$ for some integer N which is obviously not $(6 + 2\sqrt{3})^2$ or $(\frac{39+9\sqrt{13}}{2})^2$.

$(B_{12}, 2)$ corresponds to $SO(25)$ at a 50th root of unity. Since the global quantum dimension of this rank 16 category must reside in $Q[e^{\pi i/100}]$, it is clear that no integer multiple of its global quantum dimension can be $(6 + 2\sqrt{3})^2$ or $(\frac{39+9\sqrt{13}}{2})^2$.

$(C_4, 10)$ corresponds to $Sp(8)$ at a 30th root of unity, having rank 1001 [R1]. Since the quantum dimension of any simple object is ≥ 1 for a unitary theory, thus the total quantum order of any orbifold theory is at least $2\sqrt{1001} \cong 63.3$. Similarly, $(A_6, 7)$ is a unitary MTC from $SU(7)$ at a 14th root of unity. By [R1], its rank is $\binom{13}{7}$. It follows that the total quantum order of $(A_6, 7) \geq \sqrt{\binom{13}{7}}$. Hence any

nontrivial orbifold of $(A_6, 7)$ will have a total quantum order $D_G \geq 2\sqrt{\binom{13}{7}} \cong 82.8$.

But $D_{\mathcal{Z}(\mathcal{E})} = 6 + 2\sqrt{3}$, and $D_{\mathcal{Z}(\mathcal{H})} = \frac{39+9\sqrt{13}}{2} \cong 35.7$, hence it is impossible for either to be an orbifold of $(C_4, 10)$ or $(A_6, 7)$.

(3): The coset TQFT is obtained from $(D_{35})_2/(A_7)_{20}$. This embedding is as follows: the fundamental representation $\mu = \omega_4$ of $SU(8)$ is of dimension 70. ω_4 has a symmetric invariant bilinear form which gives rise to the embedding of $SU(8)$ into $SO(70)$, corresponding to the fundamental representation $\lambda = \omega_1$ of D_{35} . Hence the branching rule for λ is simply μ , and the coset theory has a simple object labeled by (λ, μ) . The embedding index can be computed using the formula above: $\chi_{\mu} = \frac{70 \cdot 18}{2 \cdot 63}$, $\chi_{\lambda} = \frac{70 \cdot 69}{2 \cdot 35 \cdot 69}$, so $\chi_{\lambda/\mu} = 10$. For level $k = 1$ of D_{35} , this embedding is conformal, i.e. the resulting coset has chiral central charge 0.

For level $k = 2$, the resulting coset has chiral central charge 24. By the formulas (18.42) on Page 805 [DMS] (cf. [Xu2]), the twist of the simple object (λ, μ) in the coset is $\frac{\theta_\lambda}{\theta_\mu}$. When $k = 2$, D_{35} corresponds to $SO(70)$ at a 70th root of unity $q = e^{\frac{\pi i}{70}}$, and the twist of λ is $\theta_\lambda = q^{(\lambda, \lambda + 2\rho)} = e^{-\frac{\pi i}{70}}$. The level for A_7 is 20 since the embedding index is 10. $So(A_7)_{20}$ corresponds to $SU(8)$ at a 28th root of unity $a = e^{\frac{\pi i}{28}}$, so the twist of μ is $\theta_\mu = a^{(\mu, \mu + 2\rho)} = e^{\frac{9\pi i}{14}}$. The twists of $\mathcal{Z}(\mathcal{E})$ are all 12th root of unity, and $\mathcal{Z}(\mathcal{H})$ all 39th root of unity. Since the ratio $\theta_{(\lambda, \mu)} = \frac{\theta_\lambda}{\theta_\mu} = e^{-\frac{23\pi i}{35}}$ can never be a 12th or 39th root of unity, hence this coset MTC is neither $\mathcal{Z}(\mathcal{E})$ nor $\mathcal{Z}(\mathcal{H})$. \square

8. APPENDIX

8.1. **Category $\frac{1}{2}E_6$.** The category $\mathcal{E} = \frac{1}{2}E_6$ is a unitary monoidal spherical category of rank 3. The following is the information for its structure. (Details can be found in [HH])

- simple objects:

$$\{1, x, y\}$$

- fusion rule:

$$x^2 = 1 + 2x + y, xy = x = yx$$

- basis:

$v_{11}^1 \in V_{11}^1$, $v_{1x}^x \in V_{1x}^x$, $v_{x1}^x \in V_{x1}^x$, $v_{1y}^y \in V_{1y}^y$, $v_{y1}^y \in V_{y1}^y$, $v_{xy}^x \in V_{xy}^x$, $v_{yx}^x \in V_{yx}^x$, $v_{yy}^1 \in V_{yy}^1$, $v_{xx}^1 \in V_{xx}^1$, $v_{xx}^y \in V_{xx}^y$, and $v_1, v_2 \in V_{xx}^x$, where V_{xy}^z denotes $\text{Hom}_{\mathcal{E}}(xy, z)$.

- associativities:

$$\begin{aligned} a_{y,y,y}^y &= a_{x,y,y}^x = a_{y,y,x}^x = a_{x,y,x}^1 = a_{x,x,y}^1 = a_{y,x,y}^y = a_{y,x,x}^1 = a_{y,x,x}^y = 1, a_{x,y,x}^y = \\ a_{y,x,y}^x &= -1, a_{x,y,x}^x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, a_{x,x,y}^x = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, a_{y,x,x}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, a_{x,x,x}^1 = \\ \frac{1}{\sqrt{2}}e^{7\pi i/12} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, a_{x,x,x}^y &= \frac{1}{\sqrt{2}}e^{7\pi i/12} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \\ a_{x,x,x}^x &= \begin{bmatrix} \frac{-1+\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} \\ \frac{-1+\sqrt{3}}{2} & \frac{1-\sqrt{3}}{2} & \frac{1-\sqrt{3}}{4}e^{\pi i/6} & \frac{1-\sqrt{3}}{4}e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4}e^{2\pi i/3} & -\frac{1-\sqrt{3}}{4}e^{\pi i/6} \\ 1 & 1 & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2}e^{5\pi i/6} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{\pi i/3} \\ 1 & 1 & \frac{1}{2}e^{\pi i/3} & \frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1) \\ 1 & -1 & -\frac{1}{2}(e^{\pi i/6}-1) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{-\pi i/3}+i) & -\frac{1}{2}e^{\pi i/3} \\ -1 & 1 & -\frac{1}{2}e^{\pi i/3} & -\frac{1}{2}(e^{-\pi i/3}+i) & \frac{1}{2}e^{5\pi i/6} & -\frac{1}{2}(e^{\pi i/6}-1) \end{bmatrix} \end{aligned}$$

- notations for the dual basis:

we use $v_z^{xy} \in \text{Hom}_{\mathcal{E}}(z, xy)$ to denote the dual basis of v_{xy}^z in the sense that $v_{xy}^z \circ v_z^{xy} = id_z$, and use v^1 and v^2 for dual bases of v_1 and v_2 , respectively.

- rigidity:

$$d_y := v_{yy}^1, b_y := v_1^{yy}, d_x := v_{xx}^1, b_x := (1 + \sqrt{3})v_1^{xx}.$$

- quantum dimensions:

$$\dim_{\mathcal{E}}(1) = 1, \dim_{\mathcal{E}}(y) = 1, \dim_{\mathcal{E}}(x) = 1 + \sqrt{3}.$$

8.2. Definitions and Lemmas. In this section, we follow Section 3 of [M1].

Definition 8.1. Let C be a strict monoidal category and let $x \in C$. A half braiding e_x for x is a family $\{e_x(y) \in \text{Hom}_C(xy, yx), y \in C\}$ of isomorphisms satisfying

- (i) *Naturality:* $f \otimes id_x \circ e_x(y) = e_x(z) \circ id_x \otimes f \quad \forall f : y \rightarrow z$.
- (ii) *The braid relation:* $e_x(y \otimes z) = id_y \otimes e_x(z) \circ e_x(y) \otimes id_z \quad \forall y, z \in C$.
- (iii) *Unit property:* $e_x(1) = id_x$.

The following lemma is equivalent to Lemma 3.3 of [M1].

Lemma 8.2. Let C be semisimple and $\{x_i, i \in \Gamma\}$ a basis of simple objects. Let $z \in C$. Then there is a one-to-one correspondence between (i) families of morphisms $\{e_z(x_i) \in \text{Hom}_C(zx_i, x_i z), i \in \Gamma\}$ such that

$$e_z(x_k) \circ id_z \otimes f = f \otimes id_z \circ id_{x_i} \otimes e_z(x_j) \circ e_z(x_i) \otimes id_{x_j} \quad \forall i, j, k \in \Gamma, f \in \text{Hom}_C(x_i x_j, x_k),$$

and (ii) families of morphisms $\{e_z(x) \in \text{Hom}_C(zx, xz, x \in C\}$ satisfying 1. and 2. from the Definition 8.1. All $e_z(x), x \in C$ are isomorphisms iff all $e_z(x_i), i \in \Gamma$ are isomorphisms.

Definition 8.3. The quantum double $\mathcal{Z}(C)$ of a strict monoidal category C has as objects pairs (x, e_x) , where $x \in C$ and e_x is a half braiding. The morphisms are given by

$$\text{Hom}_{\mathcal{Z}(C)}((x, e_x), (y, e_y)) = \{f \in \text{Hom}_C(x, y) \mid id_z \otimes f \circ e_x(z) = e_y(z) \circ f \otimes id_z \quad \forall z \in C\}.$$

The tensor product of objects is given by $(x, e_x) \otimes (y, e_y) = (xy, e_{xy})$, where $e_{xy}(z) = e_x(z) \otimes id_y \circ id_x \otimes e_y(z)$.

The tensor unit is $(1, e_1)$ where $e_1(x) = id_x$. The composition and tensor product of morphisms are inherited from C . The braiding is given by

$$c((x, e_x), (y, e_y)) = e_x(y).$$

8.3. Solutions for the half braiding. For any object $x, y \in \mathcal{E}$, $\text{Hom}_{\mathcal{E}}(xy, yx)$ has a basis consisting of morphisms of the type $(v_{x_k}^{yx})_j \circ (v_{xy}^{xk})_i$ where $k \in \Gamma$ and $1 \leq i, j \leq \dim(\text{Hom}_{\mathcal{E}}(xy, x_k))$. We parameterize each half braiding as a linear combination of such basis vectors and need to determine the coefficients satisfying all constraints in Definition 8.1. However, from Lemma 8.2, we only need to consider naturality with respect to the basis morphisms in Section 8.1. The following are the solutions where $x \in \mathcal{E}$ has 5 half braidings denoted by $e_{x_i}, i = 1, 2, \dots, 5$:

- $e_y(y) = -v_1^{yy} \circ v_{yy}^1$
- $e_y(x) = iv_x^{xy} \circ v_{yx}^x$
- $e_{x_1}(y) = iv_x^{yx} \circ v_{xy}^x$
- $e_{x_1}(x) = iv_1^{xx} \circ v_{xx}^1 + v_y^{xx} \circ v_{xx}^y + e^{-\pi i/3} v^1 \circ v_1 + e^{-5\pi i/6} v^2 \circ v_2$
- $e_{x_2}(y) = iv_x^{yx} \circ v_{xy}^x$

$$\begin{aligned}
-- \quad e_{x_2}(x) &= e^{-5\pi i/6} v_1^{xx} \circ v_{xx}^1 + e^{2\pi i/3} v_y^{xx} \circ v_{xx}^y + \frac{1-\sqrt{3}}{2} v^1 \circ v_1 + \left(\frac{\sqrt{3}}{2}\right)^{1/2} i v^2 \circ v_1 + \\
&\left(\frac{\sqrt{3}}{2}\right)^{1/2} v^1 \circ v_2 + \frac{\sqrt{3}-1}{2} i v^2 \circ v_2 \\
-- \quad e_{x_3}(y) &= i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{x_3}(x) &= e^{-5\pi i/6} v_1^{xx} \circ v_{xx}^1 + e^{2\pi i/3} v_y^{xx} \circ v_{xx}^y + \frac{1-\sqrt{3}}{2} v^1 \circ v_1 - \left(\frac{\sqrt{3}}{2}\right)^{1/2} i v^2 \circ v_1 - \\
&\left(\frac{\sqrt{3}}{2}\right)^{1/2} v^1 \circ v_2 + \frac{\sqrt{3}-1}{2} i v^2 \circ v_2 \\
-- \quad e_{x_4}(y) &= -i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{x_4}(x) &= e^{-\pi i/3} v_1^{xx} \circ v_{xx}^1 + e^{\pi i/6} v_y^{xx} \circ v_{xx}^y + \frac{1}{\sqrt{2}} e^{\pi i/4} v^1 \circ v_1 + \frac{1}{\sqrt{2}} e^{-\pi i/4} v^2 \circ v_1 + \\
&\frac{1}{\sqrt{2}} e^{\pi i/4} v^1 \circ v_2 + \frac{1}{\sqrt{2}} e^{3\pi i/4} v^2 \circ v_2 \\
-- \quad e_{x_5}(y) &= -i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{x_5}(x) &= e^{2\pi i/3} v_1^{xx} \circ v_{xx}^1 + e^{-5\pi i/6} v_y^{xx} \circ v_{xx}^y + \frac{1}{\sqrt{2}} e^{-3\pi i/4} v^1 \circ v_1 + \frac{1}{\sqrt{2}} e^{-\pi i/4} v^2 \circ \\
&v_1 + \frac{1}{\sqrt{2}} e^{\pi i/4} v^1 \circ v_2 + \frac{1}{\sqrt{2}} e^{-\pi i/4} v^2 \circ v_2 \\
-- \quad e_{1+x}(y) &= v_y^{y1} \circ v_{1y}^y - i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{1+x}(x) &= (-2 + \sqrt{3}) v_x^{x1} \circ v_{1x}^x + (2\sqrt{3} - 3) v^1 \circ v_{1x}^x + (2\sqrt{3} - 3) v^2 \circ v_{1x}^x + \\
&e^{-5\pi i/6} v_x^{x1} \circ v_1 + e^{-\pi i/3} v_x^{x1} \circ v_2 + v_{xx}^{xx} \circ v_{xx}^1 + i v_{xx}^{xx} \circ v_{xx}^y + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-5\pi i/12} v^1 \circ v_1 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{3\pi i/4} v^2 \circ \\
&v_1 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-3\pi i/4} v^1 \circ v_2 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{\pi i/12} v^2 \circ v_2 \\
-- \quad e_{y+x}(y) &= -v_{yy}^{yy} \circ v_{yy}^1 - i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{y+x}(x) &= (-2 + \sqrt{3}) i v_x^{xy} \circ v_{yx}^x + (2\sqrt{3} - 3) i v^1 \circ v_{yx}^x - (2\sqrt{3} - 3) i v^2 \circ v_{yx}^x + \\
&e^{\pi i/6} v_{xx}^{yx} \circ v_1 + e^{-\pi i/3} v_{xx}^{yx} \circ v_2 - v_{xx}^{xx} \circ v_{xx}^1 - i v_{xx}^{xx} \circ v_{xx}^y + \frac{\sqrt{3}-1}{\sqrt{2}} e^{7\pi i/12} v^1 \circ v_1 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{3\pi i/4} v^2 \circ \\
&v_1 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-3\pi i/4} v^1 \circ v_2 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-11\pi i/12} v^2 \circ v_2 \\
-- \quad e_{1+y+x}(y) &= -v_y^{y1} \circ v_{1y}^y + v_{yy}^{yy} \circ v_{yy}^1 + i v_x^{yx} \circ v_{xy}^x \\
-- \quad e_{1+y+x}(x) &= 2e^{-5\pi i/6} v_{xx}^{xy} \circ v_{1x}^x + 2e^{-5\pi i/6} v^1 \circ v_{1x}^x + 2e^{\pi i/6} v^2 \circ v_{1x}^x + \frac{2-\sqrt{3}}{2} e^{5\pi i/6} v_x^{x1} \circ \\
&v_{yx}^x + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-\pi i/4} v^1 \circ v_{yx}^x + \frac{\sqrt{3}-1}{\sqrt{2}} e^{-\pi i/4} v^2 \circ v_{yx}^x + v_{xx}^{xx} \circ v_{xx}^1 - i v_{xx}^{xx} \circ v_{xx}^y + \frac{\sqrt{3}-1}{4} i v_x^{x1} \circ \\
&v_1 + \frac{\sqrt{3}-1}{4} v_x^{x1} \circ v_2 + \frac{1}{\sqrt{2}} e^{-7\pi i/12} v_{xx}^{xy} \circ v_1 + \frac{1}{\sqrt{2}} e^{-\pi i/12} v_{xx}^{xy} \circ v_2 + \frac{\sqrt{3}-1}{\sqrt{2}} e^{5\pi i/12} v^1 \circ v_1 + \\
&\frac{\sqrt{3}-1}{\sqrt{2}} e^{-\pi i/12} v^2 \circ v_2
\end{aligned}$$

We will use the following notation:

$\mathbf{1} := (1, e_1)$, $Y := (y, e_y)$, $X_i := (x, e_{x_i})$ for $i = 1, 2, \dots, 5$, $U := (1 + x, e_{1+x})$, $V := (y + x, e_{y+x})$, and $W := (1 + y + x, e_{1+y+x})$.

It is not hard to see that all these 10 objects are simple in the quantum doubled category $\mathcal{Z}(\mathcal{E})$, and not isomorphic to each other by considering each $\text{Hom}_{\mathcal{Z}(\mathcal{E})}$ -space in Definition 8.3. Furthermore, these 10 objects completes the list of representatives of isomorphism classes of simple objects in $\mathcal{Z}(\mathcal{E})$ by the fact $\dim \mathcal{Z}(\mathcal{E}) = (\dim \mathcal{E})^2$ (see Theorem 4.14 of [M1]).

To decompose each tensor product into direct sum of simple objects, we need to compute fusion morphisms satisfying the conditions in Definition 8.3. After parameterizing each morphism as a linear combination of basis morphisms in 8.1,

to find solutions for each coefficient is purely algebraic computation, from which we can determine the dimension of each $\text{Hom}_{\mathbb{Z}(\varepsilon)}$ -space. This can be done easily.

REFERENCES

- [AH] M. Asaeda; U. Haagerup, *Exotic subfactors of finite depth with Jones indices $(5 + \sqrt{13})/2$ and $(5 + \sqrt{17})/2$* . Comm. Math. Phys. 202 (1999), no. 1, 1–63.
- [Ba] P. Bantay, *The kernel of the modular representation and the Galois action in RCFT*. Comm. Math. Phys. 233 (2003), no. 3, 423–438.
- [BB] F. Bais; P. Bouwknegt, *A classification of subgroup truncations of the bosonic string*, Nucl. Phys. B 279, 561(1987).
- [BM] D. Belov; G. Moore, *Classification of spin abelian Chern-Simons theories*, hep-th/0505235.
- [BK] B. Bakalov; A. Kirillov, Jr., *Lectures on Tensor Categories and Modular Functors*, University Lecture Series, vol. **21**, Amer. Math. Soc., 2001.
- [BEK] J. Bökenhauer; D. Evans; Y. Kawahigashi, *Longo-Rehren subfactors arising from α -induction*. Publ. Res. Inst. Math. Sci. 37 (2001), no. 1, 1–35.
- [DFNSS] S. Das Sarma; M. Freedman; C. Nayak; S. H. Simon,.; A. Stern, *Non-Abelian Anyons and Topological Quantum Computation*, arXiv:0707.1889.
- [DMS] P. Di Francesco; P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer; 1st ed. 1997. Corr. 2nd printing edition (January 18, 1999).
- [DW] R. Dijkgraaf; E. Witten, *Topological gauge theories and group cohomology*. Comm. Math. Phys. 129 (1990), no. 2, 393–429.
- [E] D. E. Evans, *From Ising to Haagerup*. Markov Process. Related Fields 13 (2007), no. 2, 267–287.
- [EG] P. Etingof; S. Gelaki, *Some properties of finite dimensional semisimple hopf algebras*. Math. Res. Lett. 5 (1998), no. 1-2, 191–197.
- [EK] D. Evans; Y. Kawahigashi, *Quantum symmetries on operator algebras*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.
- [EP1] D. Evans; P. Pinto, *Modular invariants and their fusion rules*. Advances in quantum dynamics (South Hadley, MA, 2002), 119–130, Contemp. Math., 335, Amer. Math. Soc., Providence, RI, 2003
- [EP2] D. Evans; P. Pinto, *Modular invariants and the double of the Haagerup subfactor*. Advances in operator algebras and mathematical physics, 67–88, Theta Ser. Adv. Math., 5, Theta, Bucharest, 2005.
- [Feit1] W. Feit, *The representation theory of finite groups*. North-Holland Mathematical Library, 25. North-Holland Publishing Co., Amsterdam-New York, 1982.
- [Feit2] W. Feit, *On finite linear groups in dimension at most 10*. Proceedings of the Conference on Finite Groups (Univ. Utah, Park City, Utah, 1975), pp. 397–407. Academic Press, New York, 1976.
- [FKLW] M. Freedman; A. Kitaev; M. Larsen; Z. Wang, *Topological quantum computation*. Bull. Amer. Math. Soc. (N.S.) 40 (2003), no. 1, 31–38.
- [Fin] M. Finkelberg, *An equivalence of fusion categories*. Geom. Funct. Anal. 6 (1996), no. 2, 249–267
- [FQ] D. Freed; F. Quinn, *Chern-Simons theory with finite gauge group*. Comm. Math. Phys. 156 (1993), no. 3, 435–472.
- [GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.4.10; 2007. (<http://www.gap-system.org>)

- [GW] R. Goodman; N. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge University Press, Cambridge, 1998.
- [HH] T. Hagge; S. Hong, *Some non-braided fusion categories of rank 3*, arXiv: 0704.0208
- [Hu] Y. -Z. Huang, *Vertex operator algebras, the Verlinde conjecture, and modular tensor categories*. Proc. Natl. Acad. Sci. USA 102 (2005), no. 15, 5352–5356 (electronic).
- [Iz] M. Izumi, *The structure of sectors associated with Longo-Rehren inclusions. II. Examples*. Rev. Math. Phys. 13 (2001), no. 5, 603–674.
- [Jo] V. Jones, *Index for subfactors*. Invent. Math. 72 (1983), no. 1, 1–25.
- [K] C. Kassel, *Quantum Groups*. Graduate Texts in Mathematics, 155. Springer-Verlag, New York, 1995.
- [Ki1] A. Kitaev, *Fault-tolerant quantum computation by anyons*. Ann. Physics 303 (2003), no. 1, 2–30.
- [Ki2] A. Kitaev, *Anyons in an exactly solved model and beyond*. Ann. Physics 321 (2006), no. 1, 2–111.
- [KL] Y. Kawahigashi; R. Longo, *Classification of local conformal nets: Case $c < 1$* , Ann. Math. 160 (2004), 493–522.
- [KLM] Y. Kawahigashi; R. Longo; M. Müger, *Multi-interval subfactors and modularity of representations in conformal field theory*, Comm. Math. Phys. 219 (2001), 613–669.
- [LRW] M. J. Larsen; E. C. Rowell; Z. Wang, *The N -eigenvalue problem and two applications*, Int. Math. Res. Not. **2005** (2005), no. 64, 3987–4018.
- [Ma] M. Manoliu, *Abelian Chern-Simons theory. I. A topological quantum field theory*. J. Math. Phys. 39 (1998), no. 1, 170–206.
- [M1] M. Müger, *From subfactor to categories and topology, II* J. Pure Appl. Algebra 180 (2003), no. 1-2, 159–219.
- [M2] M. Müger, *On the structure of modular categories*, Proc. London Math. Soc. (3) 87 (2003), no. 2, 291–308.
- [M3] M. Müger, *Conformal orbifold theories and braided crossed G -categories*. Comm. Math. Phys. 260 (2005), no. 3, 727–762.
- [O1] V. Ostrik, *Pre-modular categories of rank 3*, arXiv:math/0503564.
- [O2] V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*. Transform. Groups 8 (2003), no. 2, 177–206.
- [MS1] G. Moore; N. Seiberg, *Lectures on RCFT*. Superstrings '89 (Trieste, 1989), 1–129, World Sci. Publ., River Edge, NJ, 1990.
- [MS2] G. Moore; N. Seiberg, *Classical and quantum conformal field theory*. Comm. Math. Phys. 123 (1989), no. 2, 177–254.
- [Re] K. -H. Rehren, *Braid group statistics and their superselection rules*. The algebraic theory of superselection sectors (Palermo, 1989), 333–355, World Sci. Publ., River Edge, NJ, 1990.
- [R1] E. C. Rowell *From quantum groups to unitary modular tensor categories* in Contemp. Math. **413** (2006), 215–230.
- [R2] E. C. Rowell *On a family of non-unitarizable ribbon categories*, Math. Z. 250 (2005), no. 4, 745–774.
- [R3] E. C. Rowell, *Unitarizability of premodular categories* to appear in J. Pure Appl. Algebra. arXiv: 0710.1621.
- [RSW] E. Rowell; R. Stong; Z. Wang, *On classification of modular tensor categories*, math.QA/0712.1377.
- [RT] N. Reshetikhin; V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*. Invent. Math. 103 (1991), no. 3, 547–597.

- [Sc] A. N. Schellekens, *Meromorphic $c = 24$ conformal field theories*, Comm. Math. Phys. Volume 153, Number 1 (1993), 159–185.
- [Su] J. Sunday, *Presentations of the groups $SL(2, m)$ and $PSL(2, m)$* . Canad. J. Math. 24 (1972), 1129–1131.
- [SW] K. Suzuki; M. Wakui, *On the Turaev-Viro-Ocneanu invariant of 3-manifolds derived from the E_6 -subfactor*. Kyushu J. Math. 56 (2002), no. 1, 59–81.
- [Tu] V. Turaev, *Quantum Invariants of Knots and 3-Manifolds*, De Gruyter Studies in Mathematics, Walter de Gruyter (July 1994).
- [TV] V. Turaev and O. Viro, *State sum invariants of 3-manifolds and quantum $6j$ -symbols*. Topology 31 (1992), no. 4, 865–902.
- [TW] I. Tuba; H. Wenzl, *On braided tensor categories of type BCD* J. reine angew. Math. **581** (2005), 31–69.
- [Xu1] F. Xu, *Mirror extensions of local nets*. Comm. Math. Phys. 270 (2007), no. 3, 835–847.
- [Xu2] F. Xu, *3-manifold invariants from cosets*. J. Knot Theory Ramifications 14 (2005), no. 1, 21–90.
- [Xu3] F. Xu, *Some computations in the cyclic permutations of completely rational nets*. Comm. Math. Phys. 267 (2006), no. 3, 757–782.
- [Wa] Z. Wang, *Topologization of electron liquids with Chern-Simons theory and quantum computation*. Differential geometry and physics, 106–120, Nankai Tracts Math., 10, World Sci. Publ., Hackensack, NJ, 2006.
- [We] H. Wenzl, *C^* tensor categories from quantum groups*. J. Amer. Math. Soc. 11 (1998), no. 2, 261–282.
- [Wil] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific Pub Co Inc (December 1990).
- [Witt1] E. Witten, *Quantum field theory and the Jones polynomial*. Comm. Math. Phys. 121 (1989), no. 3, 351–399.
- [Witt2] E. Witten, *The central charge in three dimensions*. Physics and mathematics of strings, 530–559, World Sci. Publ., Teaneck, NJ, 1990.
- [Witt3] E. Witten, *The search for higher symmetry in string theory*. Physics and mathematics of strings. Philos. Trans. Roy. Soc. London Ser. A 329 (1989), no. 1605, 349–357.

E-mail address: seuhong@indiana.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN 47405, U.S.A.

E-mail address: rowell@math.tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, U.S.A.

E-mail address: zhenghwa@microsoft.com

MICROSOFT STATION Q, ELINGS HALL 2237, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, U.S.A.